ON THE STABILITY OF THE SOLUTIONS OF A SYSTEM OF TWO FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS FOR THE RESONANCE CASE

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Many works, not mentioned in this note, have been devoted to the investigation of the stability of solutions of systems of two first-order differential equations with periodic coefficients. In the present work there is given, on the basis of [1], a criterion of the stability of the solutions in the most difficult resonance case.

Let us consider a system of linear differential equations of the form

$$\frac{dy}{dt} = (A + \mu B(t))y \qquad (A = \text{const}). \tag{1}$$

Here y is a two-dimensional vector, μ is a small parameter ($\mu \geqslant 0$), A and $B(t)$ are real 2×2 matrices

$$B(t) = \sum_{k=-\infty}^{\infty} B_k e^{ikt}, \qquad \sum_{k=-\infty}^{\infty} |B_k| \leqslant c_1, \qquad |A| \leqslant c_2 \tag{2}$$

Here the B_k are constant complex matrices. The symbol $|A|$ stands for the norm of the matrix $A = \| a_{sj} \|_1^2$, where we assume that

$$|A| = \max\{|a_{11}| + |a_{12}|, \ |a_{21}| + |a_{22}|\} \tag{3}$$

Let us suppose that the characteristic exponents p_1 and p_2 of the solutions of the system of differential equations

$$\frac{dy}{dt} = Ay \tag{4}$$

are numbers of the form $\mp 0.5\, ni$ ($i = 1, 2, 3, \ldots$).

Then the fundamental matrix of the solutions of the system (4) will have the form [2, p. 99]

$$\exp\{At\} = C_{n1}\, e^{nit/2} + C_{n2}\, e^{-nit/2} \tag{5}$$

The complex-adjoint matrices C_{n1}, C_{n2} can be represented in the following way:

$$C_{n1}=0.5\, E - in^{-1}A, \qquad C_{n2} = 0.5E + in^{-1}A \tag{6}$$

Here, and in the sequel, E is the unit matrix. Let us make the substitution $y = \exp\{At\}\, z$ in the system of equations (1). Then we obtain

$$\frac{dz}{dt} = \mu D\,(t)\, z, \qquad D\,(t) = e^{-At}\, B\,(t)\, e^{At} \tag{7}$$

where

$$D\,(t) = \sum_{k=-\infty}^{\infty} D_k\, e^{ikt}, \qquad D_k = C_{n1}B_kC_{n1} + C_{n2}B_kC_{n2} + C_{n2}B_{k-n}C_{n1} + C_{n1}B_{k+n}C_{n2} \tag{8}$$

The problems on the stability of the solutions of the systems (1) and (7) are equivalent.

In [1] it is shown that the characteristic exponents p_1 and p_2 of the solutions of the system of the differential equations (7) are the roots, which vanish when $\mu = 0$, of the transcendental equation

$$\operatorname{Det}\Big(Ep - \mu D_0 - \tag{9}$$

$$- \sum_{\sigma=2}^{\infty} \mu^{\sigma} \sum_{\varkappa} D_{k_1}(E\,(p - k_1 i) - \mu D_0)^{-1} D_{k_2}\,(E\,(p - (k_1 + k_2)i) - \mu D_0)^{-1} \ldots$$

$$\ldots D_{k_{\sigma-1}}\,(E\,(p - (k_1 + k_2 \ldots + k_{\sigma-1})i) - \mu D_0)^{-1} D_{k_\sigma}\Big) = 0$$

$$\varkappa = (k_1 + k_2 + \ldots + k_\sigma = 0, \quad k_j \neq 0,$$

$$(j=1, \ldots, \sigma);\ 0 \overline{\in} \{k_1,\ \ k_1 + k_2 + \ldots,\ k_1 + k_2 + \ldots + \ k_{\sigma-1}\})$$

Here p is a complex variable varying in some given finite region. The series in (9) converges for sufficiently small values of μ. (In [1] a general method is proposed which makes it possible to express the matrix series in (9) in terms of a finite number of series which converge for any finite value of μ when $p \in \Sigma$.)

$$\chi_1 = - S_p B_0 = - S_p D_0$$

$$\chi_2 = \operatorname{Det}\Big(D_0 + \sum_{\sigma=2}^{\infty} (-\mu)^{\sigma-1} \sum_{\varkappa} D_{k_1}\,(Ek_1 i + \mu D_0)\Big)^{-1} D_{k_2}\,(E\,(k_1 + k_2)\, i + \mu D_0)^{-1} \ldots \tag{10}$$

$$D_{k_{\sigma-1}}(E(k_1+k_2+\ldots+k_{\sigma-1})\,i+\mu D_0)^{-1}D_{k_\sigma}\Big)$$

$$\varkappa=(k_1+k_2+\ldots+k_\sigma=0, \quad k_j\neq 0.$$

$$(j=1,2,\ldots,\sigma;\quad 0\subseteqq\{k_1,\ k_1+k_2,\ \ldots,\ k_1+k_2+\ldots+k_{\sigma-1}\})$$

Here χ_1 and χ_2 are real numbers.

Since the matrix $D(t)$ in (7) is real on the boundary of the region of instability in the space of the parametric coefficients of the system of equations (1), one of the characteristic exponents p_1 or p_2 is zero, i.e. $\chi_2 = 0$.

Expanding Equation (9) for small enough values $|p|$, μ, we obtain

$$p^2+\mu\chi_1 p+\mu^2\chi_2+0(|p\mu^2|+|\mu^3|)=0 \tag{11}$$

From the Routh-Hurwitz theorem [2, p. 433] we deduce the following result.

Theorem. Let $\mu > 0$ be a sufficiently small number.

1. If $\chi_1 > 0$, $\chi_2 > 0$, then the solutions of the system (1) are asymptotically stable.

2. If $\chi_1 < 0$, or $\chi_2 < 0$, then the solutions of the system (1) are unstable.

3. If $\chi_1 > 0$, $\chi_2 = 0$, then the solutions of the system (1) are stable, and there exists one periodic solution when n is even, or one semi-periodic solution when n is odd. The period of these solutions is 2π.

4. If $\chi_1 = 0$, $\chi_2 > 0$, then the solutions of the system (1) are stable (bounded).

5. If $\chi_1 = 0$, $\chi_2 = 0$, then the question regarding the stability requires further investigation. In this case the characteristic exponents of the solution of the system (1) are numbers of the form $\pm\ 0.5\ ni$.

Note 1. Since $k_j \neq 0$ in the series (10), the norm of the series which determines χ_2 in (10) is dominated by a geometric progression of ratio q

$$q=\mu(E-\mu|D_0|)^{-1}\sum_{k=-\infty,k\neq 0}^{\infty}|D_k| \tag{12}$$

From (12), (8), (6), (2) it follows that the condition $|q| < 1$ will be satisfied if

$$0\leqslant\mu\leqslant\Big(\sum_{k=-\infty}^{\infty}|D_k|\Big)^{-1}\leqslant\frac{n^2}{c_1(n+2c_2)^2} \tag{13}$$

If condition (13) is satisfied then the series (10) will converge.

Note 2. The quantity $\mu > 0$ must be sufficiently small in order that the characteristic exponents p_1 and p_2 of the solutions of the system (1) may not take on values $\pm(0.5n \pm 1)i$, i.e. in order that the parametric coefficients of the system (1) may not fall into a neighboring region of instability. In this case the series for χ_2 in (10) may converge.

Example. Let us evaluate χ_2 ($\chi_1 = 0$) for Mathieu's equation with $n = 2$

$$\frac{d^2x}{dt^2} + (a + 2\mu \cos 2t)\, x = 0 \tag{14}$$

Let us assume that $a \approx 1$, $b \approx 0$, $\mu = 1$. Setting $y_1 = x$, $y_2 = dx/dt$, and writing Equation (14) in the form (1), we obtain

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 0 & 0 \\ 1-a & 0 \end{pmatrix}, \qquad B_2 = B_{-2} = \begin{pmatrix} 0 & 0 \\ -b & 0 \end{pmatrix} \tag{15}$$
$$C_{21} = 0.5\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \qquad C_{22} = 0.5\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

Indicating the adjoint by a bar over a letter, we obtain the next formula from (8):

$$D_0 = \frac{1}{2}\begin{pmatrix} 0 & a-1-b \\ 1-a-b & 0 \end{pmatrix}, \quad D_2 = \bar{D}_{-2} = \frac{1}{4}\begin{pmatrix} i(1-a) & 1-a+2b \\ 1-a-2b & -i(1-a) \end{pmatrix}$$
$$D_4 = \bar{D}_{-4} = \frac{b}{4}\begin{pmatrix} -i & -1 \\ -1 & i \end{pmatrix} \tag{16}$$

Retaining in the series for χ_2 in (10) only the terms for which $\sigma = 2$, and setting $(Eki + D_0)^{-1} \approx -ik^{-1}E$, we obtain

$$\chi_2 = \operatorname{Det}\left(D_0 - \frac{1}{2i} D_2 D_{-2} + \frac{1}{2i} D_{-2} D_2 - \frac{1}{4i} D_4 D_{-4} + \frac{1}{4i} D_{-4} D_4 + \ldots\right)$$
$$= 0.25\,[(a - 1 - 0.25\,(a-1)^2 - 0.125b^2)^2 - (b - 0.5b\,(1-a))^2 + \ldots] \tag{17}$$

The equation $\chi_2 = 0$ gives the approximate equation of the boundaries of the region of instability

$$a = 1 \pm b - \frac{1}{8} b^2 + \ldots \tag{18}$$

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